Gauge theory, calibrated geometry and harmonic spinors

Andriy Haydys
University of Bielefeld

October 22, 2009

Abstract

In this paper connections between different gauge-theoretical problems in high and low dimensions are established. In particular it is shown that higher dimensional asd equations on total spaces of spinor bundles over low dimensional manifolds can be interpreted as Taubes-Pidstrygach's generalization of the Seiberg-Witten equations. By collapsing each fibre of the spinor bundle to a point, solutions of the Taubes-Pidstrygach equations are related to generalized harmonic spinors. This approach is also generalized for arbitrary fibrations (without singular fibres) compatible with an appropriate calibration.

1 Introduction

The basic set-up of low dimensional gauge theory have been generalized to higher dimensions by Donaldson and Thomas [DT]. Such generalisation requires a suitable geometric structure on the base higher dimensional manifold, e.g. a metric with holonomy G_2, $Spin(7)$ or $SU(n)$. In this paper we consider gauge theory based on the $Spin(7)$-structure mainly because of applications we keep in mind, but we could equally well start with G_2– or $SU(n)$–structures.

A Riemannian eight-manifold W with holonomy $Spin(7)$ is endowed with a particular closed 4-form Ω called Cayley form, which is a calibration [HL]. Vice versa, a Cayley calibration determines a metric with holonomy $Spin(7)$. The 4-form Ω induces the splitting of $\Lambda^2 T^*W$ into two subbundles $\Lambda^2_+ T^*W$ and $\Lambda^2_- T^*W$ of ranks 7 and 21 respectively. Then we say that a connection A is anti-self-dual (or that A is an instanton) if the self-dual part of the curvature F_A^+ vanishes.

The Ω-asd equations have a lot in common with the celebrated four-dimensional asd equations. For instance, both are first order elliptic PDEs such that solutions are points of the absolute minimum of the Yang-Mills functional. There are important distinctions, though. Whereas in dimension four asd equations are defined on any (smooth oriented Riemannian) manifold, in higher dimensions base manifolds must carry a very special geometric structure (in our case metric with holonomy in $Spin(7)$). Also, the geometric structure of the base manifold is much more involved in the compactification of the moduli space of instantons for higher dimensional manifolds [Tia]. An interplay between gauge theories in both high and low dimensions and calibrated geometry is the central aspect of this paper.

A particularly important role in our approach is played by eight-manifolds with a structure of a Cayley fibration, i.e. a map $\rho\colon W^8 \to X^4$ such that each fibre of ρ is an Ω-calibrated submanifold of W. The first nontrivial example of a Cayley fibration[1] has been constructed in [BS]. In the construction of Bryant and Salamon W is the total space of the spinor bundle over the sphere $S^4 = X$ and ρ is the natural projection. If X is an arbitrary spin manifold, then on the total space of the spinor bundle the Cayley 4-form Ω still exists, but it is not closed in general. Nevertheless, the asd condition still makes sense and gives rise to an elliptic problem.

One motivation to study the asd equations on the total spaces of spinor bundles over general four-manifolds is as follows. Recall that the key ingredients in Tian's construction of the compactified moduli space of higher dimensional instantons are (possibly singular) calibrated submanifolds. Assume $X^4 \hookrightarrow W^8$ is a smooth calibrated submanifold (Cayley submanifold). Then, by the result of McLean [McL], the normal bundle of X is isomorphic to the (twisted) spinor bundle of X. Therefore we hope that our computation will be useful in a detailed study of the boundary of the compactified moduli space. We refer to [DS, Section 6] for more details on this issue.

Another motivation comes from low dimensional topology. Suppose we are granted a construction that associates a $Spin(7)$-manifold W_X^8 to each smooth four-manifold X (possibly equipped with an additional structure). Then, by counting asd-instantons on W_X we should get an invariant of X. For instance, we can associate to each spin four-manifold the total space of its spinor bundle as mentioned above. Then Theorem 4.6 (the main result of this paper) essentially states that counting instantons on the spinor bundle of X is equivalent to counting Taubes-Pidstrygach monopoles [Tau2, Pid2] on X.

There are a few problems with the above approach. One is that the Cayley

[1] this was in fact the first example of a complete $Spin(7)$-manifold

form Ω on W_X is not always closed as already mentioned above. However, the theory of Ω-asd instantons can still be developed if the closedness condition of Ω is suitably weakend, see for example [Tia, Thm. 6.1.3] or [DS, Sect.3]. In any case, the closedness of Ω is *not* essential for our computations, but some sort of substitute implying a priori estimates is needed if one wishes to use convergence arguments.

Another problem is as follows. The main idea of the Taubes-Pidstrygach generalization of the Seiberg-Witten equations is to replace the fibre $\mathbb{C}^2$ of the spinor bundle with an arbitrary hyperKähler manifold M equipped with suitable symmetries. In our case M happens to be infinite-dimensional. There are however some evidences that the resulting low-dimensional gauge theory can be phrased in terms of sections of finite-dimensional vector bundles. This issue is briefly discussed in Section 6.

The paper is organized as follows. In introductory Section 2 main ideas in the simplest case of the flat base space are briefly sketched. In Section 3 some basic definitions are recalled and the construction of the Bryant-Salamon precalibration [BS] on the total space of the spinor bundle $\mathbb{W}^+$ over a four-manifold X is reviewed. The author is unaware whether the asd connections with respect to the Bryant-Salamon precalibration admit a priori estimates.

Section 4 is the core of the paper. Here we prove that the asd equations on $\mathbb{W}^+$ can be interpreted as Taubes-Pidstrygach equations for a suitable choice of the target space M (Theorem 4.6). The construction of the Taubes-Pidstrygach equations is reviewed at the beginning of Section 4.1.

Further, by collapsing each fibre of $\mathbb{W}^+$ to a point we show that there exists a bijective correspondence between the moduli space of eight dimensional instantons and the space of generalised harmonic spinors [Hay], whose target space is the moduli space of four-dimensional instantons (Corollary 4.10). This is a variant of the adiabatic limit reduction outlined in [DT]. It is worth to note that Corollary 4.10 can be proven without the Taubes-Pidstrygach construction, but in the authors opinion this statement is best understood from this more abstract point of view at least if one is interested in applications to low dimensional topology.

We do not discuss analytic aspects of the adiabatic limit in this paper because of several reasons. The major reason is that a prerequisite for the analytic part of the proof is a completion of the space of nonlinear harmonic spinors, which is yet to be constructed. This problem also arises in the recent paper [HNS].

Theorem 4.9 is a quaternionic analogue of the relation between moduli spaces of solutions to the symplectic vortex equations and pseudoholomorphic curves in symplectic reductions [CGS, CGMS, GS]. More precisely, a

four-manifold takes place of a Riemann surface, the role of the symplectic vortex equations is played by the Taubes-Pidstrygach equations, the symplectic reduction is replaced by the hyperKähler reduction, and pseudoholomorphic curves become generalized harmonic spinors. However there is an important distinction between the complex and quaternionic cases. Whereas in the complex case the focus is on the target manifold (or rather on its symplectic reduction), in the quaternionic case it is interesting to study both the target and the source with the help of the Taubes-Pidstrygach equations. Indeed, even the choice of the simplest admissible target manifold $\mathbb{H}$ leads to the standard Seiberg-Witten theory, which carries information about the smooth structure of the source manifold.

In Section 5 we show how our previous results modify in the case of an arbitrary Cayley fibration without singular fibres.

2 A toy model

In this section basic ideas in the simplest case of the flat space are outlined. Precise definitions and more details on computations are given below.

The total space of the spinor bundle of the flat four-manifold $X = \mathbb{R}^4$ is the flat space $\mathbb{R}^8 = X \times W^+$ equipped with a translation-invariant Cayley 4-form Ω. A connection A invariant with respect to the W^+-directions on the trivial G-bundle consists of a connection a on $\underline{G} \to X$ and a Higgs field Φ, which is a section of the trivial bundle with fibres $W^+ \otimes \mathfrak{g}_{\mathbb{C}}$, where $\mathfrak{g} = Lie(G)$. Donaldson and Thomas [DT] observe that A is Ω-asd iff the following equations hold

$$\begin{cases} \mathcal{D}_a \Phi = 0, \\ F_a^+ = [\Phi, \Phi^*]. \end{cases} \tag{1}$$

Here $\mathcal{D}_a$ is the Dirac operator on X coupled to the connection a, the bracket in the second equation is the combination of the Lie-bracket and the map $W^+ \otimes \overline{W}^+ \to \Lambda^2_+ T^*X$. It is worth to point out the striking similarity between equations (1) and the renowned Seiberg-Witten equations.

It turns out that the above observation of Donaldson and Thomas fits into a much wider picture, which is based on the following two statements. The first statement is that each (not necessarily translation invariant) $Spin(7)$-instanton on $\mathbb{R}^4 \times W^+$ is a solution of certain generalized Seiberg-Witten equations. The second one is that this conclusion still holds if $\mathbb{R}^4 \times W^+$ is replaced by the spinor bundle of an arbitrary four-manifold X. In the rest of this section the first statement is explained in more details. The second one is explained in the subsequent sections.

Let V denote the tangent space of $X = \mathbb{R}^4$ at a fixed point (the origin, say). Observe that both V and W^+ are equipped with the quaternionic structures and therefore we can naturally identify both $\Lambda^2_+ V^*$ and $\Lambda^2_+(W^+)^*$ with $\mathbb{R}^3$. A 2-form ω on $X \times W^+$ can be decomposed into its Künneth-type components: $\omega = \omega_{2,0} + \omega_{1,1} + \omega_{0,2}$. Think of $\omega_{p,q}$ as a function on $\mathbb{R}^8$ with values in $\Lambda^p V^* \otimes \Lambda^q (W^+)^*$. Then ω is Ω-asd iff

$$Cl(\omega_{1,1}) = 0 \quad \text{and} \quad \omega^+_{2,0} = \omega^+_{0,2}, \tag{2}$$

where $Cl\colon V^* \otimes (W^+)^* \cong V \otimes W^+ \to W^-$ is the standard Clifford multiplication in dimension four.

Decompose a connection A on the trivial G-bundle over $X \times W^+$ into its Künneth-type components:

$$A = a + b, \quad a \in C^\infty\big(\mathbb{R}^8;\, V^* \otimes \mathfrak{g}\big),\ b \in C^\infty\big(\mathbb{R}^8;\, (W^+)^* \otimes \mathfrak{g}\big).$$

Then from (2) it follows that A is Ω-asd iff the following equations hold

$$\begin{cases} Cl(d_{W^+}a + d_X b + [a,b]) = 0, \\ F_a^+ = F_b^+, \end{cases} \tag{3}$$

where one can think of a and b as families of connections on trivial G-bundles over X and W^+ respectively.

Putting away $Spin(7)$-instantons for a while, we describe the generalization of the Seiberg-Witten equations due to Taubes [Tau2] and Pidstrygach [Pid2] in the case of the flat four-manifold. Let M (called the target space in the sequel) be a hyperKähler manifold with a tri-Hamiltonian action of a Lie group $\mathcal{G}$. For $\xi \in Lie(\mathcal{G})$ denote by K_ξ the corresponding Killing vector field. Let a be a connection on the trivial $\mathcal{G}$-bundle over X. For a section $u \in \Gamma(\underline{M}) \cong C^\infty(X;\, M)$ we define the covariant derivative $\nabla^a u \in \Omega^1(X;\, u^*TM) \cong C^\infty(X;\, V^* \otimes u^*TM)$ by $\nabla^a u = du - K_a(u)$. Recall that with suitable identifications the Clifford multiplication is the map $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to \mathbb{H}$, $x \otimes y \mapsto \bar{x}y$. Since TM carries a natural action of quaternions, we get a variant of the Clifford multiplication $Cl\colon V^* \otimes u^*TM \to u^*TM$. The following equations for a pair (a,u)

$$\begin{cases} \mathcal{D}_a u = Cl(\nabla^a u) = 0, \\ F_a^+ + \mu(u) = 0, \end{cases}$$

are called Taubes-Pidstrygach equations, where $\mu\colon M \to Lie(\mathcal{G}) \otimes \mathbb{R}^3$ is the moment map of the $\mathcal{G}$-action.

Consider the special case $M = \Omega^1(W^+;\mathfrak{g})$, where the hyperKähler structure on $\Omega^1(W^+;\mathfrak{g})$ is induced from the flat hyperKähler structure of W^+. Then the action of the gauge group $\mathcal{G} = C^\infty(W^+;\, G)$ is tri-Hamiltonian with the moment map $\mu(b) = F_b^+$. Finally, a straightforward computation shows that for our choice of the target space the Taubes-Pidstrygach equations for a pair $(a,b) \in \Omega^1(X;\, Lie(\mathcal{G})) \times C^\infty(X;\, \Omega^1(W^+;\, \mathfrak{g})) \cong C^\infty(\mathbb{R}^8;\, V^* \otimes \mathfrak{g}) \times C^\infty(\mathbb{R}^8;\, (W^+)^* \otimes \mathfrak{g})$ are exactly equations (3). This proves the first statement.

3 Preliminaries

3.1 Differential forms on fibre bundles

Let X be a manifold and H be a Lie group. Let M be another manifold endowed with an action of the group H. Pick a principal H-bundle $\pi : Q \to X$ with a connection φ and denote by $\mathbb{M} \xrightarrow{\rho} X$ the associated fibre bundle: $\mathbb{M} = Q \times_H M$.

The connection φ determines a splitting $\mathrm{T}\mathbb{M} = \mathcal{H}_{\mathbb{M}} \oplus \mathcal{V}_{\mathbb{M}}$ into horizontal and vertical subbundles and therefore we get

$$\Omega^k(\mathbb{M}) = \bigoplus_{p+q=k} \Omega^{p,q}(\mathbb{M}), \quad \text{where} \;\; \Omega^{p,q}(\mathbb{M}) = \Gamma(\Lambda^p \mathcal{H}^*_{\mathbb{M}} \otimes \Lambda^q \mathcal{V}^*_{\mathbb{M}}).$$

We will often make use of the following construction called a "change of fibre" in the sequel. The infinite dimensional graded vector space $\Omega(M)$ inherits an action of H and therefore we have the associated vector bundle $\mathcal{E} \to X$ of infinite rank: $\mathcal{E} = Q \times_H \Omega(M)$. One can think of $\mathcal{E}$ as the fibre bundle obtained by replacing each fibre $\mathbb{M}_x \cong M$ by $\Omega(\mathbb{M}_x)$.

The connection φ induces the covariant derivative $\nabla^\varphi \colon \Gamma(\mathcal{E}) \to \Omega^1(\mathcal{E})$, which extends to the map $d_\varphi \colon \Omega^p(\mathcal{E}) \to \Omega^{p+1}(\mathcal{E})$. Identifying $\Omega^p(\mathcal{E}^q)$ with $\Omega^{p,q}(\mathbb{M})$ we see that $d_\varphi \colon \Omega(\mathbb{M}) \to \Omega(\mathbb{M})$ is a homomorphism of bidegree $(1,0)$. On the other hand, the exterior derivative $d \colon \Omega(M) \to \Omega(M)$ is H-invariant and therefore induces a homomorphism $d_v \colon \Omega(\mathbb{M}) \to \Omega(\mathbb{M})$ of bidegree $(0,1)$. It turns out that unlike in the case of Cartesian product, the exterior derivative on $\mathbb{M}$ has one more component, which we describe next.

Let K_ξ denote the Killing vector field of the H-action on M corresponding to $\xi \in \mathfrak{h} = Lie(H)$. The contraction $\mathfrak{h} \otimes \Omega(M) \to \Omega(M)$, $\xi \otimes \omega \mapsto \imath_{K_\xi}\omega$ defines a homomorphism of vector bundles $ad\, Q \otimes \mathcal{E} \to \mathcal{E}$. Then the curvature form Φ of the connection φ induces the map $\imath_\Phi \colon \Omega^p(\mathcal{E}^q) \to \Omega^{p+2}(\mathcal{E}^{q-1})$ via the combination of wedging and contraction.

Theorem 3.1 ([BL]). *The exterior derivative $d_{\mathbb{M}}\colon \Omega(\mathbb{M}) \to \Omega(\mathbb{M})$ decomposes as follows: $d_{\mathbb{M}} = d_v + d_\varphi - \imath_\Phi$.*

Let $P \to M$ be a principal G–bundle. We assume that a lift of the H-action to P is provided such that the actions of G and H commute. Then the associated bundle $\mathbb{P} = Q \times_H P$ yields a principal G-bundle over $\mathbb{M}$:

$$\begin{array}{ccc} Q \times P & \longrightarrow & \mathbb{P} \\ \downarrow & & \downarrow \\ Q \times M & \longrightarrow & \mathbb{M}. \end{array}$$

Denote by $\mathcal{E}^0(ad\, P)$ (respectively $\mathbb{A}$) the fibre bundle obtained by replacing each fibre $\mathbb{M}_x$ by $\Omega^0(\mathbb{M}_x;\, ad\,\mathbb{P})$ (respectively $\mathcal{A}(i_x^*\mathbb{P})$), where $i_x\colon \mathbb{M}_x \hookrightarrow \mathbb{M}$ is the inclusion. The connection φ determines the covariant derivatives on both $\mathcal{E}^0(ad\, P)$ and $\mathbb{A}$ as well as the inclusions $\hat{\cdot}\colon \Gamma(\mathcal{E}^0(ad\, P)) \hookrightarrow \Gamma(ad\,\mathbb{P})$ and $\hat{\cdot}\colon \Gamma(\mathbb{A}) \hookrightarrow \mathcal{A}(\mathbb{P})$. For any $a \in \Gamma(\mathbb{A}), b \in \Omega^1(\mathcal{E}^0(ad\, P))$ the sum $A = \hat{a} + \hat{b}$ is a connection on $\mathbb{P}$. Vice versa, any connection A on $\mathbb{P}$ can be decomposed as $\hat{a} + \hat{b}$ for some a, b as above.

Proposition 3.2. *For a connection $A = \hat{a} + \hat{b}$ the components of the curvature $F_A \in \Omega^2(\mathbb{M}; ad\,\mathbb{P})$ are given by the following formulae:*

$$F_A^{0,2} = F_a; \tag{4}$$
$$F_A^{1,1} = \nabla^\varphi a + \nabla^a b; \tag{5}$$
$$F_A^{2,0} = -\imath_\Phi a + d_\varphi b + [b, b]. \tag{6}$$

The proof of the above Proposition can be obtained, for instance, by a straightforward application of Theorem 3.1 to the local representations of connection forms. We omit the details. However a few comments to formulae (4)-(6) are in place.

First notice that each fibre of the bundle $\mathcal{E}^0(ad\, P)$ is naturally a Lie algebra. Then the term $[b, b]$ in (6) means the combination of the Lie brackets and wedging. Further, for each fixed $x \in X$ the value of a at x gives a connection $\nabla^{a(x)}$ on $\Omega^0(i_x^* ad\,\mathbb{P})$. On the other hand, the value of b at x lies in $\Omega^0(i_x^* ad\,\mathbb{P}) \otimes T_x^* X$ and therefore the (vertical) covariant derivative $\nabla^{a(x)} b(x)$ is well defined. It is abbreviated as $\nabla^a b$ in (5).

3.2 The group Spin(7), some subgroups and representations

Denote by $\mathbb{H}$ the $\mathbb{R}$-algebra of quaternions and by $Sp(1)$ the group of all quaternions of unit length. The basic complex representation W of $Sp(1)$ is

given by

$$(q,x)\mapsto qx,\qquad q\in Sp(1),\ x\in\mathbb{H}\cong W. \tag{7}$$

Consider the group $K=\big(Sp(1)\times Sp(1)\times Sp(1)\big)/\pm 1$, where -1 acts componentwise. It is convenient to give a certain label to each component of K as follows

$$K=\big(Sp_+(1)\times Sp_-(1)\times Sp_0(1)\big)/\pm 1. \tag{8}$$

Let U be the real 8-dimensional K-representation given by the following action of K on $\mathbb{R}^8\cong\mathbb{H}\oplus\mathbb{H}$:

$$\kappa\cdot(x,y)=(q_+x\,\bar q_-,\ q_+y\,\bar q_0),\qquad \kappa=[q_+,q_-,q_0]\in K. \tag{9}$$

Then U is the direct sum of two real irreducible K-representations E and F such that

$$E_{\mathbb{C}}\cong W^+\otimes W^-,\quad F_{\mathbb{C}}\cong W^+\otimes W^0. \tag{10}$$

Let θ (respectively η) denote the projection of $\mathbb{R}^8=\mathbb{H}\oplus\mathbb{H}$ onto the first (resp. second) component. It is convenient to think of θ and η as $\mathbb{H}$-valued 1-forms on $\mathbb{R}^8$. The following 4-form

$$\Omega=-\frac{1}{24}\mathrm{Re}\left(\theta\wedge\bar\theta\wedge\theta\wedge\bar\theta-6\,\theta\wedge\bar\theta\wedge\eta\wedge\bar\eta+\eta\wedge\bar\eta\wedge\eta\wedge\bar\eta\right) \tag{11}$$

is K-invariant. Recall that the stabiliser of Ω in $GL_8(\mathbb{R})$ is $Spin(7)$ [BS] and therefore $K\subset Spin(7)$. Notice also that $Spin(7)$ preserves both the Euclidean metric and the orientation [BS] on $\mathbb{R}^8$, i.e. we have the following inclusions $K\subset Spin(7)\subset SO(8)$.

Think of $\mathbb{R}^8$ as a $Spin(7)$-representation via the inclusion $Spin(7)\subset SO(8)$. The linear map

$$\mathrm{T}_\Omega:\Lambda^2\big(\mathbb{R}^8\big)^*\to\Lambda^2\big(\mathbb{R}^8\big)^*,\qquad \omega\mapsto -*(\Omega\wedge\omega)$$

has two eigenvalues 3 and -1. The corresponding eigenspaces $\Lambda^2_+\big(\mathbb{R}^8\big)^*$ and $\Lambda^2_-\big(\mathbb{R}^8\big)^*$ are irreducible $Spin(7)$-representations of dimensions 7 and 21 respectively [Bry]. One can check that the collection of 2-forms $(\omega_1,\dots,\omega_7)$, where

$$\begin{aligned}\omega_1 i+\omega_2 j+\omega_3 k&=\theta\wedge\bar\theta-\eta\wedge\bar\eta,\\ \omega_4+\omega_5 i+\omega_6 j+\omega_7 k&=\bar\theta\wedge\eta,\end{aligned} \tag{12}$$

is a basis of $\Lambda^2_+\big(\mathbb{R}^8\big)^*$. Hence we have an isomorphism of K-representations

$$\Lambda^2_+U^*\cong\mathfrak{sp}_+(1)\oplus V, \tag{13}$$

where V is the standard representation of $SO(4)=(Sp_-(1)\times Sp_0(1))/\pm 1$.

Taking into account (10) it is easy to see that there is essentially a unique homomorphism $E \otimes F \to V$. Its complexification is the four-dimensional Clifford multiplication $Cl : W^+ \otimes W^+ \otimes W^- \to W^-$ twisted by W^0.

Denote by $\Pi\colon \Lambda^2_+ U^* \cong \Lambda^2 E^* \oplus E^* \otimes F^* \oplus \Lambda^2 F^* \longrightarrow \Lambda^2_+ U^*$ the natural projection. Combining the above observations we get that Π maps $E^* \otimes F^*$ onto the V-component of $\Lambda^2_+ U^*$ only and the composition $\Pi'\colon E^* \otimes F^* \to \Lambda^2_+ U^* \to V$ is the (twisted) Clifford multiplication.

On the other hand, both $\Lambda^2_+ E^*$ and $\Lambda^2_+ F^*$ are naturally isomorphic to $\mathfrak{sp}_+(1)$, which in turn is the other irreducible component of $\Lambda^2_+ U^*$. Then Π maps $\Lambda^2 E^* \oplus \Lambda^2 F^*$ onto the $\mathfrak{sp}_+(1)$-component of $\Lambda^2_+ U^*$ only and a computation shows that the composition $\Pi''\colon \Lambda^2 E^* \oplus \Lambda^2 F^* \to \Lambda^2_+ U^* \to \mathfrak{sp}_+(1)$ is given by

$$\Pi''(\alpha, \beta) = \alpha^+ - \beta^+,$$

where the above identifications are understood.

Remark 3.3. Although it is natural to take the standard Euclidean metric on $\mathbb{R}^8$, one can also consider the following perturbation

$$\begin{aligned} g_\varepsilon &= g_{\scriptscriptstyle E} \oplus \varepsilon g_{\scriptscriptstyle F} = \mathrm{Re}\left(\theta \otimes \bar\theta + \varepsilon\, \eta \otimes \bar\eta\right), \\ \Omega_\varepsilon &= -\frac{1}{24}\mathrm{Re}\left(\theta \wedge \bar\theta \wedge \theta \wedge \bar\theta - 6\varepsilon\, \theta \wedge \bar\theta \wedge \eta \wedge \bar\eta + \varepsilon^2\, \eta \wedge \bar\eta \wedge \eta \wedge \bar\eta\right) \end{aligned}$$

for $\varepsilon > 0$. Then by tracing the above computations we get

$$\Pi_\varepsilon = \Pi'_\varepsilon + \Pi''_\varepsilon, \qquad \Pi'_\varepsilon = \varepsilon \Pi', \quad \Pi''_\varepsilon(\alpha, \beta) = \alpha^+ - \varepsilon^{-1}\beta^+.$$

3.3 Spin(7)-structures on spinor bundles

For an oriented Riemannian manifold $\mathbb{W}^8$ a $Spin(7)$-structure is a principal $Spin(7)$-subbundle of the $SO(8)$-bundle of orthonormal oriented frames or, equivalently, a 4-form Ω, whose restriction to each tangent space lies in the $SO(8)$-orbit of the standard 4-form (11). In this section, following [BS], we describe a $Spin(7)$-structure on the total space of the spinor bundle over a four-manifold.

From now on X denotes a smooth closed oriented Riemannian manifold. Let $\tilde Q \xrightarrow{\tilde\pi} X$ be the $SO(4)$-principal bundle of oriented isometries $\varkappa : T_x X \to \mathbb{H}$. Denote by $\tilde\theta \in \Omega^1(\tilde Q; \mathbb{H})$ the tautological 1-form, i.e. $\tilde\theta(\mathrm{v}) = \tilde f(\tilde\pi_* \mathrm{v})$, where $\mathrm{v} \in T_{\tilde f}\tilde Q$. We also assume that X is spin and pick a $Spin(4) = Sp_+(1) \times Sp_-(1)$-structure $Q \xrightarrow{\pi} X$, which is a double cover of $\tilde Q$. The Levi-Civita connection (φ, ψ) is an equivariant 1-form on Q with values in

$\mathfrak{sp}_+(1) \oplus \mathfrak{sp}_-(1) \cong \operatorname{Im}\mathbb{H} \oplus \operatorname{Im}\mathbb{H}$. Let x be the quaternionic variable on $\mathbb{H}$. Denote $\eta = dx - \varphi x \in \Omega^1(Q \times \mathbb{H}; \mathbb{H})$ and put

$$\Omega = -\frac{1}{24}\operatorname{Re}\Big(\theta \wedge \bar\theta \wedge \theta \wedge \bar\theta - 6\,\theta \wedge \bar\theta \wedge \eta \wedge \bar\eta + \eta \wedge \bar\eta \wedge \eta \wedge \bar\eta\Big) \in \Omega^4(Q \times \mathbb{H}),$$

where θ is the pull-back of $\tilde\theta$.

Further, define an action of $Spin(4)$ on $Q \times \mathbb{H}$ by the rule $(f, x)\cdot(q_+, q_-) = (f \cdot (q_+, q_-),\ \bar q_+ x)$. Clearly, the quotient space is the positive spinor bundle $\rho\colon \mathbb{W}^+ \to X$. It is easy to check that the 4-form Ω is $Spin(4)$-invariant and basic. Therefore Ω descends to a 4-form (denoted by the same letter) on the total space of $\mathbb{W}^+$ and defines a $Spin(7)$-structure. The corresponding metric is given by $g = \operatorname{Re}\big(\theta \otimes \bar\theta + \eta \otimes \bar\eta\big)$.

Remark 3.4. One can replace the $Spin(4)$-bundle Q in the above setting by a principal K-bundle Q' such that $Q'/Sp_0(1) = \tilde Q$. In particular, we can choose an embedding $S^1 \hookrightarrow Sp_0(1)$ and take Q' as a principal $Spin^c(4) = \big(Sp_+(1) \times Sp_-(1) \times S^1\big)/\pm 1$ bundle. Therefore we could have started with an arbitrary closed smooth oriented Riemannian four-manifold, since for such manifolds a $Spin^c(4)$-structure always exists. However, in this case one needs to make a choice of connection on the determinant bundle $Q'/Spin(4)$.

On the other hand, choosing the $Spin(4)$-action differently one can obtain $Spin(7)$-structures on T^*X or $\underline{\mathbb{R}} \oplus \Lambda^2_+ T^*X$. In these cases the existence of $Spin(4)$-structures is also redundant.

4 Spin(7)-instantons and Taubes-Pidstrygach equations

4.1 ASD equations on spinor bundles as Taubes-Pidstrygach system

The main aim of this section is to prove that the ASD equations on the total space of a spinor bundle $\mathbb{W}^+ \to X$ is an example of the Taubes-Pidstrygach system. We first sketch the Taubes-Pidstrygach construction in a from suitable for our purposes. The construction involves two manifolds (*a source* and *a target*), which play different roles.

Source manifold. Let X be a four-dimensional oriented Riemannian manifold, which is referred to as a source manifold in the sequel. For the sake of simplicity we assume as before that X is spin and denote by $Q_+ = Q/Sp_-(1)$ the principal bundle of $\mathbb{W}^+$.

Let $\mathcal{G}$ be a Lie group whose Lie algebra $Lie(\mathcal{G}) = \mathcal{L}$ is indoved with an Ad–invariant scalar product. Assume a homomorphism $\alpha\colon Sp(1) \to Aut(\mathcal{G})$ is given and put $\hat{\mathcal{G}} = \mathcal{G} \rtimes Sp(1)$. Let $\hat{\pi}\colon \hat{Q} \to X$ be a principal $\hat{\mathcal{G}}$-bundle such that $\hat{Q}/\mathcal{G}$ and Q_+ are isomorphic. Moreover we restrict ourselves to the case when the $\mathcal{G}$-bundle $\hat{Q} \to \hat{Q}/\mathcal{G}$ is trivial, which is enough for our purposes. In other words, $\hat{Q} = \mathcal{G} \times Q_+$ as fibred space but considered as a $\hat{\mathcal{G}}$-bundle.

Further, the decomposition of the vector space $Lie(\hat{\mathcal{G}}) = \hat{\mathcal{L}} = \mathcal{L} + \mathfrak{sp}(1)$ is invariant with respect to $Sp(1) \hookrightarrow \hat{\mathcal{G}}$. Since we have an inclusion $i\colon Q_+ \hookrightarrow \hat{Q}$, for any connection $B \in \Omega^1(\hat{Q};\ \hat{\mathcal{L}})$ we obtain

$$i^*B = b + \varphi, \tag{14}$$

where φ is a connection on Q_+ and b is a 1-form on X with values in $\mathbb{L} = Q_+ \times_{Sp_+(1)} \mathcal{L}$. A simple computation shows $F_B = F_b + \Phi$, where $F_b = d_\varphi b + [b, b] \in \Omega^2(X;\ \mathbb{L})$ and $\Phi = F_\varphi \in \Omega^2(X;\ ad\,Q_+)$ is the curvature of φ. In what follows we consider only those connections B whose $\mathfrak{sp}(1)$-component φ is the Levi-Civita connection on Q_+.

Target manifold. The other ingredient of the construction is a hyperKähler manifold M called the target space. Recall that a Riemannian manifold (M, g) is called hyperKähler if g is Kähler with respect to three complex structures (I_1, I_2, I_3) satisfying the quaternionic relations. Denote by $(\omega_1, \omega_2, \omega_3)$ the corresponding Kähler 2-forms and put $\omega = \omega_1 i + \omega_2 j + \omega_3 k \in \Omega^2(M;\ \mathrm{Im}\,\mathbb{H})$.

Further, we assume that $\hat{\mathcal{G}} = \mathcal{G} \rtimes Sp(1)$ acts isometrically on M such that the following two conditions hold:

(A) the action of $\mathcal{G} \hookrightarrow \hat{\mathcal{G}}$ is tri-Hamiltonian (in particular, preserves all complex structures);

(B) the action of $Sp(1) \hookrightarrow \hat{\mathcal{G}}$ is *permuting*, i.e.

$$(L_q)^*\,\omega = q\omega\bar{q}, \qquad q \in Sp(1), \tag{15}$$

where $L_q : M \to M,\ m \mapsto qm$.

Nonlinear Dirac operator. Below we sketch a construction of a Dirac operator acting on sections of a nonlinear bundle $\rho\colon \mathbb{M} \to X$. We refer to [Hay] for more details.

Put $\mathbb{M} = \hat{Q} \times_{\hat{\mathcal{G}}} M = Q_+ \times_{Sp_+(1)} M$. For each point $(\hat{f}, m) \in \hat{Q} \times M$ we have the following exact sequence

$$0 \longrightarrow \hat{\mathcal{L}} \longrightarrow T_{\hat{f}}\hat{Q} \times T_m M \xrightarrow{\tau_*} T_{\tau(\hat{f},m)}\mathbb{M} \longrightarrow 0,$$

where $\tau\colon \hat{Q}\times M\to\mathbb{M}$ is the natural projection map. A connection B on $\hat{Q}$ can be regarded as a $\hat{\mathcal{G}}$-invariant splitting $T\hat{Q}=\hat{\mathcal{H}}\oplus\hat{\mathcal{V}}$ and induces a similar splitting $\mathrm{T}\mathbb{M}=\mathcal{H}\oplus\mathcal{V}$, where $\mathcal{V}=\tau_*(TM)=\ker\rho_*,\ \mathcal{H}=\tau_*(\hat{\mathcal{H}})=\rho^*TX$.

Further, one can think of $f\in Q_+$ as a quaternionic structure on $T_{\pi_+(f)}X\cong\hat{\mathcal{H}}_{(g,f)}\subset T_{(g,f)}\hat{Q}$ and therefore the horizontal bundle $\hat{\mathcal{H}}$ is equipped with a quaternionic structure (J_1,J_2,J_3). For the subbundle $\hat{E}^-=Hom_{\mathbb{H}}(\hat{\mathcal{H}},TM)$ of $Hom_{\mathbb{R}}(\hat{\mathcal{H}},TM)\to\hat{Q}\times M$ we have the natural complement

$$\hat{E}^+=\{A\in Hom_{\mathbb{R}}(\mathcal{H},TM)\mid\ I_1AJ_1+I_2AJ_2+I_3AJ_3=A\}\,.$$

It follows from assumptions (A) and (B) that the splitting $\hat{\mathcal{H}}^*\otimes TM=\hat{E}^-\oplus\hat{E}^+$ is $\hat{\mathcal{G}}$-invariant and therefore we obtain

$$\mathcal{H}^*\otimes\mathcal{V}=E^-\oplus E^+,\tag{16}$$

where $E^\pm\to\mathbb{M}$ is the factor of $\hat{E}^\pm\to\hat{Q}\times M$ by the $\hat{\mathcal{G}}$-action. We denote by $\mathcal{C}\colon\mathcal{H}^*\otimes\mathcal{V}\to E^-$ the projection onto the first subbundle.

Remark 4.1. In the case $M=\mathbb{H},\ \mathcal{G}=\{1\}$ we have $\mathbb{M}=\mathbb{W}^+,\ E^-=\rho^*\mathbb{W}^-$ and $\mathcal{C}\colon\rho^*T^*X\otimes\rho^*\mathbb{W}^+\to\rho^*\mathbb{W}^-$ is the usual Clifford multiplication.

Let $ev\colon\Gamma(\mathbb{M})\times X\to\mathbb{M}$ be the evaluation map. Think of sections of ev^*E^- as maps associating to each $u\in\Gamma(\mathbb{M})$ a section of $ev^*E^-|_X\cong u^*E^-$.

Definition 4.2. The following section of ev^*E^-

$$\mathcal{D}_b\colon\Gamma(\mathbb{M})\xrightarrow{\nabla^B}\Gamma(T^*X\otimes u^*\mathcal{V})\cong\Gamma(u^*(\mathcal{H}^*\otimes\mathcal{V}))\xrightarrow{\mathcal{C}}\Gamma(u^*E^-)$$

is called a (generalized) *Dirac operator*, where B is given by (14).

Examples of generalized Dirac operators and corresponding harmonic spinors (i.e. solutions of the equation $\mathcal{D}_b\,u=0$) can be found in [Hay]. Below we present an example of a Dirac operator with an infinite-dimensional target space.

Remark 4.3. A particular role in the sequel is played by Dirac operators $\mathcal{D}=\mathcal{D}_0$ corresponding to $\mathcal{G}=\{1\}$ (and hence $b=0$). In this case, $\mathcal{D}$ is called a *pure* Dirac operator.

Below it will be useful to rewrite the equation $\mathcal{D}_b\,u=0$ in the equivariant setup as follows. Recall that a section u of $\mathbb{M}\to X$ can be identified with an equivariant map $\hat{u}\colon\hat{Q}\to M$. Then the covariant derivative ∇^Bu is given by the restriction $\hat{u}_*^h$ of the differential $\hat{u}_*$ to the horizontal subspace. It is clear from the above description that u is harmonic iff

$$I_1\hat{u}_*^hJ_1+I_2\hat{u}_*^hJ_2+I_3\hat{u}_*^hJ_3=\hat{u}_*^h,\tag{17}$$

or, equivalently, iff pointwise $\hat{u}_*^h$ has no $\mathbb{H}$–linear component.

Taubes-Pidstrygach equations. Recall that the action of $\mathcal{G} \hookrightarrow \hat{\mathcal{G}}$ on M is tri-Hamiltonian and let $\mu\colon M \to \operatorname{Im}\mathbb{H} \otimes \mathcal{L} \cong \mathfrak{sp}(1) \otimes \mathcal{L}$ be a momentum map. We also assume that μ is $Sp(1)$-equivariant, where $\mathcal{L}$ is considered as the $Sp(1)$-representation. Then the map $id \times \mu : Q_+ \times M \to Q_+ \times (\mathfrak{sp}(1) \otimes \mathcal{L})$ can be identified with a section $\nu \in \Gamma\big(\mathbb{M};\, \rho^*\big(\Lambda^2_+ X \otimes \mathbb{L}\big)\big)$. Finally, to any spinor $u \in \Gamma(\mathbb{M})$ we associate a self-dual 2-form $\nu \circ u \in \Omega^2_+(X;\, \mathbb{L})$.

Definition 4.4. The following system of first order PDEs

$$\begin{cases} \mathcal{D}_b\, u = 0, & (18) \\ F_b^+ + \nu \circ u = 0, & (19) \end{cases}$$

for a pair $(u, b) \in \Gamma(\mathbb{M}) \times \Omega^1(X;\, \mathbb{L})$ is called Taubes-Pidstrygach equations.

Denote $\mathbb{G} = P_+ \times_{Sp_+(1)} \mathcal{G}$. The gauge group $\Gamma(\mathbb{G})$ acts on the configuration space $\Gamma(\mathbb{M}) \times \Omega^1(X; \mathbb{L})$

$$g \cdot (u, b) = \big(g \cdot u,\ Ad_g\, b - (\nabla^{\varphi} g) g^{-1}\big) \tag{20}$$

and preserves the space of solutions of (18),(19). We denote by $\mathcal{M}_{TP}$ the corresponding moduli space.

We wish to take the infinite dimensional space of all connections on a principal G-bundle $P \to \mathbb{R}^4$ as the target space. First let us introduce some notation.

We assume that the group $Sp(1)$ acts on the total space of P commuting with G and descending to basic action (7) on $\mathbb{R}^4$. We also assume that P is equipped with a framing at infinity compatible with the $Sp(1)$-action. Let $\mathcal{A}^0(P)$ and $\mathcal{G}^0(P)$ consist of connections and gauge transformations on P respectively with a suitable asymptotic behaviour at infinity (see [Ito] for details). One can think of $\mathcal{A}^0(P)$ and $\mathcal{G}^0(P)$ as the space of connections and the *based* gauge group on S^4 respectively.

Put $M = W^+$ in the set-up of Section 3.1 and consider the bundle $\mathbb{P} = Q_+ \times_{Sp_+(1)} P \to \mathbb{W}^+$. For the principal bundle Q_+ denote by $\mathbb{A}, \mathbb{G}, \mathbb{L}$ the associated fibre bundles over X with fibres $\mathcal{A}^0(P), \mathcal{G}^0(P), \mathcal{L}^0(P) = Lie(\mathcal{G}^0(P))$ respectively. One can think of $\mathbb{A}, \mathbb{G}$ and $\mathbb{L}$ as the bundles obtained by replacing each fibre $\mathbb{W}^+_x$ by $\mathcal{A}^0(i_x^*\mathbb{P}), \mathcal{G}^0(i_x^*\mathbb{P})$ and $ad\,(i_x^*\mathbb{P})$ respectively.

Example 4.5 (The Dirac operator for the target $\mathcal{A}^0(P)$)**.** The construction of the Dirac operator in the case of a flat target space is somewhat simpler and we can give a more direct description.

We first choose $\Omega^1(\mathbb{R}^4)$ as the target space. The basic action (7) induces the permuting action of $Sp(1)$. Then $\Omega^1(\mathbb{R}^4)$ is isomorphic to $C^\infty(W) \otimes W$ as an $Sp(1)$-representation and we get a variant of the Clifford multiplication

$$Cl\colon W^+ \otimes W^- \otimes \Omega^1(W^+) \longrightarrow C^\infty(W^+) \otimes W^-,$$

which differs from the standard Clifford multiplication just by tensoring with $C^\infty(W^+)$. With these choices the space of spinors is $\Gamma\big(\mathcal{E}^1(\mathbb{W}^+)\big)$ and the corresponding pure Dirac operator is given by the sequence

$$\Gamma\big(\mathcal{E}^1\big(\mathbb{W}^+\big)\big) \xrightarrow{\nabla^\varphi} \Gamma\big(T^*X \otimes \mathcal{E}^1\big(\mathbb{W}^+\big)\big) \xrightarrow{Cl} \Gamma\big(\mathbb{W}^- \otimes \mathcal{E}^0(\mathbb{W}^+)\big) \cong \Gamma\big(\rho^*\mathbb{W}^-\big).$$

It follows from (13) that $\Lambda^2_+\mathrm{T}^*\mathbb{W}^+ \cong \rho^*\Lambda^2_+T^*X \oplus \rho^*\mathbb{W}^-$. Then using the identification $\Omega^p(\mathcal{E}^q) \cong \Omega^{p,q}(\mathbb{W}^+)$ as in Section 3.1, the above sequence yields:

$$\Omega^{0,1}(\mathbb{W}^+) \xrightarrow{d^{1,0}} \Omega^{1,1}(\mathbb{W}^+) \xrightarrow{\Pi'} \Omega^2_+(\mathbb{W}^+),$$

where Π' is the natural projection.

Now let us take $\mathcal{A}^0(P)$ instead of $\Omega^1(\mathbb{R}^4)$ as the target space. The gauge group $\mathcal{G} = \mathcal{G}^0(P)$ acts on $\mathcal{A}^0(P)$ preserving the hyperKähler structure. If we define the homomorphism $\alpha\colon Sp(1) \to Aut(\mathcal{G})$ by

$$(qg)(p) = g(\bar{q}\,p), \qquad p \in P, \quad g \in \mathcal{G},$$

then $\mathcal{G}^0(P) \rtimes Sp(1)$ acts on $\mathcal{A}^0(P)$ such that conditions (A) and (B) are satisfied. The corresponding Dirac operator $\mathcal{D}_b$ is given by

$$\Gamma(\mathbb{A}) \xrightarrow{\nabla^B} \Omega^1(X; \mathcal{E}^1(ad\,P)) \cong \Omega^{1,1}(\mathbb{W}^+;\, ad\,\mathbb{P}) \longrightarrow \Omega^2_+(\mathbb{W}^+;\, ad\,\mathbb{P}).$$

Further, for any $a \in \Gamma(\mathbb{A})$ we have $\nabla^B a = \nabla^\varphi a + \nabla^a b$, so that with the help of formula (5) we finally obtain

$$\mathcal{D}_b(a) = (\nabla^\varphi a + \nabla^a b)^+ = (F_A^{1,1})^+, \qquad \text{where } A = \hat{a} + \hat{b}.$$

It is well known that the moment map of the $\mathcal{G} = \mathcal{G}^0(P)$-action on $M = \mathcal{A}^0(P)$ is given by $\mu(a) = F_a^+ \in \Omega^2_+(\mathbb{R}^4;\, ad\,P) \cong \mathrm{Im}\,\mathbb{H} \otimes \mathcal{L}$. Thus we get the corresponding section ν.

Theorem 4.6. *A connection $A = \hat{a} + \hat{b}$ on $\mathbb{P} \to \mathbb{W}^+$ is anti-self-dual iff the pair $(a, b) \in \Gamma(\mathbb{A}) \times \Omega^1(X;\, \mathbb{L})$ is a solution to the Taubes-Pidstrygach-type equations with the target manifold $M = \mathcal{A}^0(P)$*

$$\begin{cases} \mathcal{D}_b\, a = 0, & (21) \\ F_b^+ + \nu \circ a = -(\imath_\Phi a)^+, & (22) \end{cases}$$

where $\Phi \in \Omega^2(X;\, \Lambda^2_+ X)$ is the curvature form of the component φ of the Levi-Civita connection.

Proof. It follows from the results of Section 3.2 that a connection A is Ω-asd iff $(F_A^{1,1})^+ = 0$ and $(F_A^{2,0} + F_A^{0,2})^+ = 0$. Recalling formulae (4) and (6) one immediately sees that the second equation is equivalent to (22). On the other hand, we have shown in Example 4.5 that the first equation is equivalent to (21). □

We note in passing that Ω_ε-asd equations (see Remark 3.3) lead to the following perturbation of equations (21),(22):

$$\begin{cases} \mathcal{D}_{b_\varepsilon} a_\varepsilon = 0, \\ \varepsilon(F_{b_\varepsilon} + \imath_\Phi a_\varepsilon)^+ + \nu \circ a_\varepsilon = 0. \end{cases}$$

4.2 Perturbed Taubes-Pidstrygach equations and formal limiting system

Consider the following perturbation of the Taubes-Pidstrygach system

$$\begin{cases} \mathcal{D}_{b_\varepsilon} u_\varepsilon = 0, & (23) \\ \varepsilon F_{b_\varepsilon}^+ + \nu \circ u_\varepsilon = 0, & (24) \end{cases}$$

where ε is a positive parameter. In this section we consider solutions $(u_\varepsilon, b_\varepsilon)$ tending to a finite limit (u_0, b_0) as $\varepsilon \to +0$, i.e. when bubbling off does not happen. Putting formally $\varepsilon = 0$ in system (23),(24) we get that (u_0, b_0) is a solution of the system

$$\begin{cases} \mathcal{D}_{b_0} u_0 = 0, & (25) \\ \nu \circ u_0 = 0. & (26) \end{cases}$$

Let $\mathcal{M}_{TP}^0$ denote the moduli space of solutions to the above system.

Remark 4.7. Perturbation similar to (23),(24) for the classical Seiberg-Witten equations was studied by Taubes [Tau3, Tau1]. It is also interesting to observe that putting formally $\varepsilon = +\infty$ we obtain the system

$$\begin{cases} \mathcal{D}_{b_\infty} u_\infty = 0, \\ F_{b_\infty}^+ = 0, \end{cases}$$

which was studied by Pidstrygach and Tyurin [PT] in the case of the linear Dirac operator.

From now on we assume that $0 \in \mathcal{L}$ is a regular value of the momentum map μ and that the group $\mathcal{G}$ acts freely on $\mu^{-1}(0)$. This assumption is crucial for the following Proposition. Further, denote by $M_0 = \mu^{-1}(0)/\mathcal{G}$

the hyperKähler reduction of M. Then the permuting action of $Sp(1)$ on M induces a permuting action on M_0 and we denote by $\mathbb{M}_0 \to X$ the associated bundle $Q_+ \times_{Sp_+(1)} M_0$. Observe also that the projection $\mu^{-1}(0) \to M_0$ gives rise to the fibrewise map $\nu^{-1}(0) \to \mathbb{M}_0$.

Proposition 4.8. *Let $0 \in \mathcal{L} \otimes \operatorname{Im}\mathbb{H}$ be a regular value of the momentum map μ and let $\mathcal{G}$ act freely on $\mu^{-1}(0)$. Pick a spinor $u \in \Gamma(\mathbb{M})$ such that $\nu \circ u = 0$ and denote by $v \in \Gamma(\mathbb{M}_0)$ its projection. Then v is a pure harmonic spinor iff there exists $b \in \Omega^1(X; \mathbb{L})$ such that $\mathcal{D}_b\, u = 0$.*

Proof. Let $\hat{u}$ and $\hat{v}$ be equivariant maps representing u and v respectively such that the following diagram

$$\begin{tikzcd} Q_+ \arrow[r, "\hat{u}"] \arrow[dr, "\hat{v}"'] & \mu^{-1}(0) \arrow[d] \\ & M_0 \end{tikzcd} \tag{27}$$

commutes. Pick a point $f \in Q_+$ and denote $m = v(f) \in \mu^{-1}(0) \subset M$. Let $\mathcal{K}_m \cong \mathcal{L}$ be the vector space spanned by Killing vectors at the point m. Define the subspace $\mathcal{H}_m \subset T_m M$ by the following orthogonal decomposition

$$T_m M = \mathcal{H}_m \oplus \mathcal{K}_m \oplus I_1\mathcal{K}_m \oplus I_2\mathcal{K}_m \oplus I_3\mathcal{K}_m.$$

Notice that $T_m\mu^{-1}(0) = \mathcal{H}_m \oplus \mathcal{K}_m$ and $T_{[m]}M_0$ can be identified with $\mathcal{H}_m$.

The image of $\hat{u}_* : T_f Q_+ \to T_m M$ is contained in $\mathcal{H}_m \oplus \mathcal{K}_m$ and the projection to $\mathcal{H}_m$ yields the differential of $\hat{v}$. Since $\mathcal{G}$ acts freely on $\mu^{-1}(0)$, for each $\mathrm{v} \in T_f Q_+$ there exists a unique $b(\mathrm{v}) \in \mathcal{L}$ such that

$$\hat{u}_*(\mathrm{v}) - \hat{v}_*(\mathrm{v}) = -K_{b(\mathrm{v})}(m). \tag{28}$$

Then $b \in \Omega^1(Q_+; \mathcal{L})$ is basic. Further, for $q \in Sp_+(1)$ denote $R_q \colon Q_+ \to Q_+,\ R_q(f) = f \cdot q$. We have

$$K_{b((R_q)_*\mathrm{v})}(\bar{q}\,m) = (L_{\bar{q}})_* K_{b(\mathrm{v})}(m) = K_{Ad_{\bar{q}}\, b(\mathrm{v})}(\bar{q}\,m),$$

where the first equality follows from (28) and the $Sp(1)$-equivariance of both $\hat{u}$ and $\hat{v}$. Since the action of $\mathcal{G}$ is free we get $\big(R_q\big)^* b = Ad_{\bar{q}}\, b$, i.e. b descends to a 1-form on X with values in $\mathbb{L}$.

Let B be the connection on $\hat{Q}$ determined by the Levi-Civita connection and the 1-form b as in (14). Then the covariant derivative of $\hat{u}$ with respect to B can be identified with the restriction of $\hat{u}_* + K_{b(\cdot)}(\hat{u})$ to the horizontal bundle $\mathcal{H}^+ \to Q_+$ of the Levi-Civita connection. It remains to note that by virtue of equation (28) v is a pure harmonic spinor iff for each point $f \in Q_+$ the restriction of the $\mathbb{R}$-linear map $\hat{u}_* + K_{b(\cdot)}(\hat{u})$ to $\mathcal{H}^+_f$ has no $\mathbb{H}$-linear component, i.e. $\mathcal{D}_b\, u = 0$. $\square$

By assumption $\mu^{-1}(0) \to M_0$ is a principal $\mathcal{G}$-bundle and therefore for a given map $\hat{v}$ the existence of a commutative diagram (27) is equivalent to the triviality of the $\mathcal{G}$-bundle $\hat{v}^*\mu^{-1}(0) \to Q_+$. Consider the following space

$$\Gamma_0(\mathbb{M}_0) = \big\{ v \in \Gamma(\mathbb{M}_0) \,|\; \hat{v}^*\mu^{-1}(0) \to Q_+ \text{ is trivial} \big\}$$

and denote by $\mathcal{H}_0(\mathbb{M}_0) \subset \Gamma_0(\mathbb{M}_0)$ the subspace of harmonic spinors.

Theorem 4.9. *Let $0 \in \mathcal{L} \otimes \operatorname{Im}\mathbb{H}$ be a regular value of the momentum map μ and let $\mathcal{G}$ act freely on $\mu^{-1}(0)$. Then there exists a one-to-one correspondence between the moduli space $\mathcal{M}^0_{TP}$ of solutions to limiting problem (25),(26) and the space $\mathcal{H}_0(\mathbb{M}_0)$ of harmonic spinors.*

Proof. It follows from Proposition 4.8 that we have a map from the space of solutions of (25),(26) to $\mathcal{H}_0(\mathbb{M}_0)$, which factors through $\mathcal{M}^0_{TP}$. To construct the inverse map pick a harmonic spinor $v \in \Gamma_0(\mathbb{M}_0)$. Then $u, u' \in \Gamma(\mathbb{M})$ satisfying $\nu \circ u = 0 = \nu \circ u'$ are lifts of v iff there exists $g \in \Gamma(\mathbb{G})$ such that $u' = g \cdot u$. Then it is easy to check that for the corresponding 1-forms b and b' as in Proposition 4.8 we have $b' = Ad_g\, b - (\nabla^\varphi g)g^{-1}$ and the statement follows. ∎

Theorem 4.9 was independently discovered by Pidstrygach [Pid1].

Observe that the hyperKähler reduction of $\mathcal{A}^0(P)$ with respect to the based gauge group action is the moduli space $\mathcal{M}_{asd}$ of framed asd-instantons. We denote $\mathbb{M}_{asd} = Q_+ \times_{Sp_+(1)} \mathcal{M}_{asd}$. One can think of $\mathbb{M}_{asd} \to X$ as the fibre bundle obtained by replacing each fibre $\mathbb{W}^+_x \cong \mathbb{R}^4$ of the usual spinor bundle $\mathbb{W}^+ \to X$ by the moduli space of instantons over $\mathbb{W}^+_x$.

On the other hand, the formal limiting form of the Ω_ε-asd equations (see Remark 3.3) is

$$\begin{cases} (F^{1,1}_{A_0})^+ = 0, \\ (F^{0,2}_{A_0})^+ = 0, \end{cases} \iff \begin{cases} \mathcal{D}_{b_0}\, a_0 = 0, \\ F^+_{a_0} = 0, \end{cases} \qquad A_0 = \hat{a}_0 + \hat{b}_0. \tag{29}$$

Corollary 4.10. *There exits a natural bijective correspondence between the moduli space of solutions to equations (29) and the subspace $\mathcal{H}_0(\mathbb{M}_{asd})$ of harmonic spinors.* ∎

Remark 4.11. When this paper has been essentially ready for publication, S. Donaldson communicated to the author a direct proof of Corollary 4.10, which has appeared in [DS, Thm 1]. The reader can also find there, among other things, the relevance of Corollary 4.10 to the compactification of the moduli space of higher dimensional instantons.

Example 4.12. Let X^4 be a compact hyperKähler manifold, so that Q_+ is flat. Then [Hay] harmonic spinors are exactly aholomorphic maps $X \to \mathcal{M}_{asd}$. It follows from [Hay, Cor. 2] that each aholomorphic map $X \to \mathcal{M}_{asd}$ is constant. Hence we have $\mathcal{H}(\mathbb{M}_{asd}) = \mathcal{M}_{asd}$.

5 Instantons and Cayley fibrations

In this section we show that a suitable modification of Corollary 4.10 holds for $Spin(7)$-manifolds equipped with a structure of Cayley fibration. It is convenient to fix some terminology first.

Definition 5.1. Let $E \to W$ be a real vector bundle of rank $4k$ over a manifold W. We say that E is *quasiquaternionic*, if a subbundle $\mathcal{I} \subset End(E)$ of real rank 3 admitting local trivializations (I_1, I_2, I_3) with quaternionic relations

$$I_1 I_2 = -I_2 I_1 = I_3, \qquad I_1^2 = I_2^2 = I_3^2 = -id \tag{30}$$

is given. In this case $\mathcal{I}$ is called the structural bundle.

Since any two trivializations of $\mathcal{I}$ as above differ by an $SO(3)$-gauge, the structural bundle is naturally an oriented Euclidean vector bundle.

An example of quasiquaternionic bundle is any oriented Euclidean vector bundle E of real rank 4. In this case we choose by default the structural bundle to be $\mathfrak{so}_+(E) \cong \Lambda^2_+ E$. Another example is the tangent bundle of a quaternionic Kähler manifold. The vertical bundle of the nonlinear spinor bundle $\mathbb{M}$ as in Section 4.1 is also quasiquaternionic.

Let X be an arbitrary oriented four-manifold. Suppose $\rho\colon W \to X$ is a fibre bundle such that the vertical bundle $\mathcal{V}_\rho = \ker \rho_*$ is quasiquaternionic. Assume also that W is equipped with a connection such that the horizontal bundle $\mathcal{H}_\rho$ is also quasiquaternionic. For instance, this is the case if X is Riemannian.

Suppose an isomorphism $\gamma\colon \mathcal{I}(\mathcal{H}_\rho) \to \mathcal{I}(\mathcal{V}_\rho)$ compatible with the Euclidean structures is given, i.e. for any local trivialization (I_1, I_2, I_3) of $\mathcal{I}(\mathcal{H}_\rho)$ satisfying (30) the triple $(J_1, J_2, J_3) = (\gamma(I_1), \gamma(I_2), \gamma(I_3))$ is a local trivialization of $\mathcal{I}(\mathcal{V}_\rho)$ also satisfying (30). Then the subbundles

$$\begin{aligned} E_\rho^+ &= \{A \in Hom_{\mathbb{R}}(\mathcal{H}_\rho, \mathcal{V}_\rho) \mid I_1 A J_1 + I_2 A J_2 + I_3 A J_3 = A\}, \\ E_\rho^- &= \{A \in Hom_{\mathbb{R}}(\mathcal{H}_\rho, \mathcal{V}_\rho) \mid I_1 A J_1 + I_2 A J_2 + I_3 A J_3 = -3A\} \end{aligned}$$

of $Hom_{\mathbb{R}}(\mathcal{H}_\rho, \mathcal{V}_\rho)$ are well defined and therefore we have the decomposition

$$\mathcal{H}_\rho^* \otimes \mathcal{V}_\rho = E_\rho^+ \oplus E_\rho^-.$$

Recall that for any $u \in \Gamma(X; W)$ the covariant derivative ∇u is a section of $T^*X \otimes u^*\mathcal{V}_\rho \cong u^*\big(\mathcal{H}_\rho^* \otimes \mathcal{V}_\rho\big)$.

Definition 5.2. The first order differential operator $\mathcal{D}$ defined by the sequence

$$\Gamma(X; W) \xrightarrow{\nabla} \Gamma(T^*X \otimes u^*\mathcal{V}_\rho) \longrightarrow \Gamma(u^*E_\rho^-)$$

is called a (generalized) *Dirac operator*.

Notice that unlike in Definition 4.2, in the above definition the source manifold X does not need to be equipped with a Riemannian structure.

From now on we assume that (W, g, Ω) is a $Spin(7)$-manifold equipped with a Cayley fibration $\rho\colon W \to X$, where X is an arbitrary oriented four-manifold. Moreover, we also assume that W is compact and ρ has no critical points. The compactness of W is assumed for the simplicity of exposition[2], whereas the second assumption is an oversimplification. However, it is a necessary step before considering the general situation when some fibres are allowed to be singular.

For the Cayley fibration we define the horizontal bundle as the orthogonal complement of $\mathcal{V}_\rho$:

$$TW = \mathcal{H}_\rho \oplus \mathcal{V}_\rho. \tag{31}$$

Observe that we have a distinguished isomorphism $\gamma\colon \mathfrak{so}_+(\mathcal{H}_\rho) \to \mathfrak{so}_+(\mathcal{V}_\rho)$. Indeed, let $Q(W) \to W$ be the $Spin(7)$-structure of W. Recall [HL, Thm 1.38] that at each point $w \in W$ the subgroup $K \subset Spin(7)$ that respects splitting (31) is isomorphic to (8) and denote by $Q_\rho \to W$ the corresponding principal K-subbundle of $Q(W)$. Then $\mathcal{H}_\rho = Q_\rho \times_K E$ and $\mathcal{V}_\rho = Q_\rho \times_K F$ for some K-representations E and F such that $\Lambda^2_+ E \cong \mathfrak{so}_+(3) \cong \Lambda^2_+ F$. Hence, we get the desired isomorphism γ.

Remark 5.3. Each fibre W_x of the Cayley fibration has a hyperHermitian structure (defined up to an $SO(3)$-rotation). Indeed, pick a frame in T_xX. Then the horizontal lift combined with the Gram-Schmidt process defines a trivialization of $\mathcal{H}_\rho|_{W_x}$, so that $\mathfrak{so}_+(\mathcal{H}_\rho)$ also carries a trivialization. Finally, we equip W_x with a hyperHermitian structure via the map γ.

Further, similarly as in Section 3.1 the space of differential forms on W is naturally equipped with the bigrading such that

$$\begin{aligned} d &= d^{1,0} + d^{0,1} + d^{2,-1}, \qquad && d\colon \Omega^k(W) \to \Omega^{k+1}(W), \\ \Omega &= \Omega^{4,0} + \Omega^{2,2} + \Omega^{0,4}, \qquad && \Omega \in \Omega^4(W). \end{aligned}$$

[2] the spinor bundle over a four-manifold provides a model for non-compact manifolds equipped with a Cayley fibration

For any $\varepsilon \in (0,1]$ consider the metric $g_\varepsilon = g_h + \varepsilon g_v$, where g_h and g_v are Euclidean scalar products on $\mathcal{H}_\rho$ and $\mathcal{V}_\rho$ respectively. The corresponding 4-form Ω_ε is of comass 1 but in general it does not need to be closed.

Lemma 5.4. *For any $\varepsilon \in (0,1)$ there exists a decomposition $\Omega_\varepsilon = \Omega_{1,\varepsilon} + \Omega_{2,\varepsilon}$ such that $\Omega_{1,\varepsilon}$ is closed and $\Omega_{2,\varepsilon}$ satisfies*

$$- \omega \wedge \omega \wedge \Omega_{2,\varepsilon} < |\omega|^2 vol_W \tag{32}$$

for any $\omega \in \Omega^2(W)$.

Proof. We have

$$\begin{aligned}\Omega_\varepsilon &= \Omega^{4,0} + \varepsilon\, \Omega^{2,2} + \varepsilon^2 \Omega^{0,4} \\ &= \varepsilon\, \Omega + \big((1-\varepsilon)\Omega^{4,0} - \varepsilon(1-\varepsilon)\Omega^{0,4}\big) \\ &= \Omega_{1,\varepsilon} + \Omega_{2,\varepsilon}.\end{aligned}$$

By assumption $\Omega_{1,\varepsilon} = \varepsilon\, \Omega$ is closed. Further, for any 2-form ω we have

$$\begin{aligned}-\omega \wedge \omega \wedge \Omega^{4,0} &= -\omega^{0,2} \wedge \omega^{0,2} \wedge \Omega^{4,0} \\ &= \Big(\big|\omega^{0,2}_-\big|^2 - \big|\omega^{0,2}_+\big|^2\Big) vol_W \le |\omega|^2 vol_W\end{aligned}$$

and similarly $\omega \wedge \omega \wedge \Omega^{0,4} \le |\omega|^2 vol_W$. Combining these inequalities, we obtain (32). ☐

Corollary 5.5 ([Tia, Thm. 6.1.3]). *Let G be a compact Lie group and $\mathbb{P} \xrightarrow{\eta} W$ be a principal G-bundle. Then for any $\varepsilon \in (0,1]$ there exists a natural compactification of the moduli space $\mathcal{M}^\varepsilon_{asd}(\mathbb{P})$ of Ω_ε-asd instantons.* ☐

Consider $\mathbb{P}$ as the fibre bundle over X via the map $\tau\colon \mathbb{P} \xrightarrow{\eta} W \xrightarrow{\rho} X$. Then a connection ϕ on $\mathbb{P} \to X$ induces a connection φ on $W \to X$. Indeed, think of a connection as a 1-form with values in the vertical bundle. Further, observe that $\mathcal{V}_\tau = \ker \eta_* \circ \rho_* = \eta_*^{-1}(\mathcal{V}_\rho)$. Then the connection φ is determined via the requirement that the diagram

$$\begin{array}{ccc} \mathrm{T}\mathbb{P} & \xrightarrow{\eta_*} & TW \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ \mathcal{V}_\tau & \xrightarrow{\eta_*} & \mathcal{V}_\rho \end{array}$$

commutes. We assume a choice of connection ϕ inducing connection (31) on W is made.

Remark 5.6. A connection ϕ as above does exist (but is not unique). Indeed, first notice that the space of all connections on $\mathbb{P} \to X$ inducing a given connection on $W \to X$ is convex. It is easy to check the existence of ϕ for trivial bundles. Then the existence of ϕ for nontrivial bundles can be obtained via glueing with the help of the partition of unity.

In the setup of Section 3.1 ϕ was fixed via the lift of the H-action from M to P and the choice of a connection on the principal H-bundle Q.

Assume that for each $x \in X$ the moduli space $\mathcal{M}_{asd}(i_x^*\mathbb{P})$ of asd-connections on the fibre W_x is nonsingular. Denote by $\mathbb{M}_{asd} \to X$ the fibre bundle obtained by replacing W_x by $\mathcal{M}_{asd}(i_x^*\mathbb{P})$. Similarly, the fibre bundle $\mathcal{A}_{asd}(i_x^*\mathbb{P}) \to \mathcal{M}_{asd}(i_x^*\mathbb{P})$ gives rise to the bundle $\mathbb{A}_{asd} \to \mathbb{M}_{asd}$ and the connection ϕ induces a connection on $\mathbb{M}_{asd} \to X$. Further, a hyperHermitian structure on W_x induces a hyperHermitian structure on the corresponding fibre of $\mathbb{M}_{asd}$. In particular the vertical bundle of $\mathbb{M}_{asd}$ is quasiquaternionic and there is also an induced isomorphism $\Gamma\colon \mathfrak{so}_+(\mathcal{H}_{\mathbb{M}_{asd}}) \to \mathcal{I}(\mathcal{V}_{\mathbb{M}_{asd}})$.

Further, similarly as in the case of the spinor bundle we can write the Ω_ε-asd equations in the form

$$\begin{cases} \Pi''_\varepsilon\big(F^{2,0}_{A_\varepsilon} + F^{0,2}_{A_\varepsilon}\big) = 0, \\ \big(F^{1,1}_{A_\varepsilon}\big)^+ = 0. \end{cases} \tag{33}$$

The formal limiting form of system (33) as $\varepsilon \to 0$ is

$$\begin{cases} \big(F^{0,2}_{A_0}\big)^+ = 0, \\ \big(F^{1,1}_{A_0}\big)^+ = 0. \end{cases} \tag{34}$$

Let $\Gamma_0(\mathbb{M}_{asd}) \subset \Gamma(\mathbb{M}_{asd})$ denote the subspace of all sections, which can be lifted to a section of $\mathbb{A}_{asd}$.

Theorem 5.7. *There exists a natural bijective correspondence between the moduli space of solutions to equations (34) and the space $\mathcal{H}_0(\mathbb{M}_{asd})$ of all harmonic spinors contained in $\Gamma_0(\mathbb{M}_{asd})$.* $\square$

The proof of the above theorem can be obtained by a suitable modification of the proof of Theorem 4.9. The difference is that we can not interpret equations (33) as a Taubes-Pidstrygach system, but it is easy to check directly that the arguments in the proof of Theorem 4.9 apply to the above statement as well. We omit the details.

6 Concluding remarks

Let M and M' be two hyperKähler manifolds endowed with actions of $Sp(1) \rtimes \mathcal{G}$ and $Sp(1) \rtimes \mathcal{G}'$ respectively such that $M_0 = M /\!/\!/ \mathcal{G}$ and $M'_0 = M' /\!/\!/ \mathcal{G}'$ are isomorphic as hypercomplex manifolds. Assume we are in a favourable situation when the spaces of solutions to the perturbed Taubes-Pidstrygach equations (23),(24) with targets M and M' are good approximations (in a suitable sense) of $\mathcal{H}_0(\mathbb{M}_0) \cong \mathcal{H}_0(\mathbb{M}'_0)$. Then the Taubes-Pidstrygach theories with target spaces M and M' are essentially equivalent.

Recall that the ADHM construction represents the moduli space $\mathcal{M}_{n,k}$ of framed $SU(n)$-instantons of charge k on $\mathbb{R}^4$ as the finite dimensional hyperKähler reduction. In other words, a natural candidate for M' in the context of $Spin(7)$-instantons on $\mathbb{W}^+ \to X$ is the vector space

$$M' \cong u(k) \otimes_{\mathbb{R}} W \,\oplus\, \mathbb{C}^n \otimes E \otimes W,$$

where E denote the standard complex representation of $U(k) = \mathcal{G}'$. Notice that if $k = 1$ we essentially arrive at the classical Seiberg-Witten theory. On the other hand, if we put formally $n = 0$, which corresponds to the choice of $\mathfrak{u}(k) \otimes \mathbb{H}$ as a target manifold, then we get equations (1), i.e. a four-dimensional analogue of Hitchin's theory [Hit2]. In general, the choice of M' as above leads to a mixture of both theories.

The problem is that the hyperKähler reduction M'_0 is *not* smooth (it is the Uhlenbeck compactification $\bar{\mathcal{M}}_{n,k}$ of $\mathcal{M}_{n,k}$) so that Theorem 4.9 is not applicable. One can partially overcome this difficulty as follows. Suppose X is a Kähler surface so that we can modify slightly the original Taubes-Pidstrygach equations:

$$\begin{cases} \mathcal{D}_b\, u = 0, \\ F_b^+ + \nu(u) = \xi\, \omega_X, \end{cases}$$

where ξ is a central element in $\mathfrak{g}' = \mathfrak{u}(k)$. Arguing along similar lines as in Section 4.2, we arrive at the space of harmonic spinors (in fact, (anti)holomorphic sections) with the target $\mathcal{M}(n,k) = M' /\!/\!/_{\mu=\xi} \mathcal{G}'$, which is smooth. In fact, there is [Nak] a natural holomorphic morphism[3] $\mathcal{M}(n,k) \to \bar{\mathcal{M}}_{n,k}$ and hence the corresponding map between the spaces of holomorphic sections.

We note in passing that it is also interesting to study the Taubes-Pidstrygach gauge theories based on analogues of the ADHM construction for other types of hyperKähler four-manifolds like tori [DK] or ALE spaces [KN]. Similarly, it is well-known that the moduli space of monopoles on $\mathbb{R}^3$ (the Atiyah-Hitchin manifold) can be constructed as infinite dimensional hyperKähler

[3]this morphism represents $\mathcal{M}(n,k)$ as a series of blow-ups of $\bar{\mathcal{M}}_{n,k}$

reduction in two different ways [Hit1]. The author intends to continue his studies in the above directions.

References

[BL] Jean-Michel Bismut and John Lott. Flat vector bundles, direct images and higher real analytic torsion. *J. Amer. Math. Soc.*, 8(2):291–363, 1995.

[Bry] Robert L. Bryant. Metrics with exceptional holonomy. *Ann. Math.*, 126:525–576, 1987.

[BS] Robert L. Bryant and Simon M. Salamon. On the construction of some complete metrics with exceptional holonomy. *Duke Math. J.*, 58(3):829–850, 1989.

[CGMS] Kai Cieliebak, A. Rita Gaio, Ignasi Mundet i Riera, and Dietmar A. Salamon. The symplectic vortex equations and invariants of Hamiltonian group actions. *J. Symplectic Geom.*, 1(3):543–645, 2002.

[CGS] Kai Cieliebak, Ana Rita Gaio, and Dietmar A. Salamon. J-holomorphic curves, moment maps, and invariants of Hamiltonian group actions. *Internat. Math. Res. Notices*, (16):831–882, 2000.

[DK] S. K. Donaldson and P. B. Kronheimer. *The geometry of four-manifolds.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1990. Oxford Science Publications.

[DS] Simon Donaldson and Ed Segal. Gauge theory in higher dimensions, II. *arXiv:0902.3239*, 2009.

[DT] S. K. Donaldson and R. P. Thomas. Gauge theory in higher dimensions. In *The geometric universe (Oxford, 1996)*, pages 31–47. Oxford Univ. Press, Oxford, 1998.

[GS] Ana Rita Pires Gaio and Dietmar A. Salamon. Gromov-Witten invariants of symplectic quotients and adiabatic limits. *J. Symplectic Geom.*, 3(1):55–159, 2005.

[Hay] Andriy Haydys. Nonlinear Dirac Operator and Quaternionic Analysis. *Communications in Mathematical Physics*, 281(1):251–261, 2008.

[Hit1] N. J. Hitchin. On the construction of monopoles. *Comm. Math. Phys.*, 89(2):145–190, 1983.

[Hit2] Nigel J. Hitchin. The self-duality equations on a Riemann surface. *Proc. Lond. Math. Soc., III. Ser.*, 55:59–126, 1987.

[HL] Reese Harvey and Jr. H. Blaine Lawson. Calibrated geometries. *Acta Math.*, 148:47–157, 1982.

[HNS] Sonja Hohloch, Gregor Noetzel, and Dietmar Salamon. Hypercontact structures and Floer homology. *arXiv.org:0807.0319*, 2008.

[Ito] Mitsuhiro Itoh. Based anti-instantons and gravitational instantons. In *Geometry of manifolds (Matsumoto, 1988)*, volume 8 of *Perspect. Math.*, pages 453–475. Academic Press, Boston, MA, 1989.

[KN] Peter B. Kronheimer and Hiraku Nakajima. Yang-Mills instantons on ALE gravitational instantons. *Math. Ann.*, 288(2):263–307, 1990.

[McL] Robert C. McLean. Deformations of calibrated submanifolds. *Comm. Anal. Geom.*, 6(4):705–747, 1998.

[Nak] Hiraku Nakajima. *Lectures on Hilbert schemes of points on surfaces*, volume 18 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999.

[Pid1] V. Pidstrygach. Private Communication.

[Pid2] V. Ya. Pidstrygach. Hyper-Kähler manifolds and the Seiberg-Witten equations. *Tr. Mat. Inst. Steklova*, 246(Algebr. Geom. Metody, Svyazi i Prilozh.):263–276, 2004.

[PT] V. Ya. Pidstrigach and A. N. Tyurin. The smooth structure invariants of an algebraic surface defined by the Dirac operator. *Izv. Ross. Akad. Nauk Ser. Mat.*, 56(2):279–371, 1992.

[Tau1] Clifford H. Taubes. SW $\Rightarrow$ Gr: from the Seiberg-Witten equations to pseudo-holomorphic curves. *J. Amer. Math. Soc.*, 9(3):845–918, 1996.

[Tau2] Clifford Henry Taubes. Nonlinear generalizations of a 3-manifold's Dirac operator. In *Trends in mathematical physics (Knoxville, TN, 1998)*, volume 13 of *AMS/IP Stud. Adv. Math.*, pages 475–486. Amer. Math. Soc., Providence, RI, 1999.

[Tau3] Clifford Henry Taubes. The Seiberg-Witten equations and the Weinstein conjecture. *Geom. Topol.*, 11:2117–2202, 2007.

[Tia] Gang Tian. Gauge theory and calibrated geometry. I. *Ann. of Math. (2)*, 151(1):193–268, 2000.